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OPTIMIZATION OF THE STRESS TENSOR IN AN ELASTIC ANISOTROPIC HALF-PLANE

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We consider the problem of optimizing the components of the stress tensor and their integral characteristics. The normal and tangential forces prescribed on the boundary of the elastic anisotropic half-plane $y \geq 0$ are chosen from certain function classes of curvilinear strip type.

Bibliography: 2 titles.

We consider an elastic anisotropic plate $-\infty < x < \infty$, $y \geq 0$ of thickness h with characteristic numbers μ_1^* and μ_2^* (cf. [1]). For simplicity we shall assume that $\mu_j^* = i\mu_j$ ($\mu_j \in \mathbb{R}$, $\mu_1 < \mu_2$). On the boundary of the half-plane there are normal and tangential forces $N(x)$ and $T(x)$ respectively.

The values of the components of the stress tensor $\{\sigma_x, \sigma_y, \tau_{xy}\}$ at each point z ($\operatorname{Im} z > 0$) are defined by the formulas

$$\begin{aligned} \sigma_x &= -2\operatorname{Re}(\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)), \\ \sigma_y &= -2\operatorname{Re}(\Phi_1'(z_1) + \Phi_2'(z_2)), \\ \tau_{xy} &= -2\operatorname{Re}[i(\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2))], \quad z_j = x + \mu_j^* y. \end{aligned} \tag{1}$$

We fix a point z ($\operatorname{Im} z > 0$). In the class of bounded functions

$$|N(x)| \leq l_N \quad (x \in U \subseteq \mathbb{R}); \quad |T(x)| \leq l_T \quad (x \in V \subseteq \mathbb{R}) \tag{2}$$

we shall seek those on which the maximal values

$$\sigma_x^* = \max|\sigma_x|, \quad \sigma_y^* = \max|\sigma_y|, \quad \tau_{xy}^* = \max|\tau_{xy}| \tag{3}$$

or certain linear combinations of them are attained.

We are interested in the problems of computing the quantities

$$\sigma_x^D = \max\left|\int\limits_D \sigma_x\, dS\right|, \quad \sigma_y^D \max\left|\int\limits_D \sigma_y\, dS\right|, \quad \tau_{xy}^D = \max\left|\int\limits_D \tau_{xy}\, dX\right|, \tag{4}$$

where D is a line or a closed region of the upper half-plane.

In what follows we also consider the cases when

$$A_N(x) \leq N(x) \leq B_N(x) \ (x \in U), \quad A_T(x) \leq T(x) \leq B_T(x) \ (x \in V). \tag{5}$$

The integral representations of the functions $\Phi_j'(x_j)$ have the form [1]

$$\Phi_1'(z_1) = \frac{1}{2\pi h(\mu_2 - \mu_1)} \int_{-\infty}^{\infty} \frac{i\mu_2 N(\xi) + T(\xi)}{\xi - z_1}\, d\xi; \quad \Phi_2'(z_2) = -\frac{1}{2\pi h(\mu_2 - \mu_1)} \int_{-\infty}^{\infty} \frac{i\mu_1 N(\xi) + T(\xi)}{\xi - z_2}\, d\xi. \tag{6}$$

We can now find

$$\sigma_k = \frac{1}{\pi h(\mu_2 - \mu_1)} \int_{-\infty}^{\infty} [Q_k(\xi; z)N(\xi) + R_k(\xi; z)T(\xi)[\, d\xi, \quad k = 1, 2, 3, \tag{7}$$

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where

$$Q_k = (-1)^{k-1}\mu_1\mu_2\rho_k(\xi; z), \quad (k = 1, 2),$$
$$Q_3 = \mu_1\mu_2(\xi - x)\rho_2(\xi; z),$$
$$R_k = (-1)^k(\xi - x)\rho_k(\xi; z) \ (k = 1, 2), \quad R_3 = y\rho_1(\xi; z),$$
$$\rho_1(\xi; z) = \frac{\mu_1^2}{(\xi - x)^2 + \mu_1^2 y^2} - \frac{\mu_2^2}{(\xi - x)^2 + \mu_2^2 y^2}, \quad \sigma_1 = \sigma_x, \ \sigma_2 = \sigma_y,$$
$$\rho_2(\xi; z) = \frac{1}{(\xi - x)^2 + \mu_1^2 y^2} - \frac{1}{(\xi - x)^2 + \mu_2^2 y^2}, \quad \sigma_3 = \tau_{xy}.$$

We now introduce the function classes

$$M_N = \{N(x) \in L^\infty(U) : A_N(x) \le N(x) \le B_N(x), \ x \in U\};$$
$$M_T = \{T(x) \in L^\infty(V) : A_T(x) \le T(x) \le B_T(x), \ x \in V\},$$

in which

$$A_N(x) \le B_N(x) (\in L^\infty(U)), \ A_T(x) \le B_T(x) (\in L^\infty(V)),$$

and $L^\infty(W)$ is the Banach space of essentially bounded functions on the set $W \subseteq R$. We set

$$N_1(x) = N(x) - \frac{1}{2}[A_N(x) + B_N(x)], \ T_1(x) = T(x) - \frac{1}{2}[A_T(x) + B_T(x)].$$

It is clear that

$$|N_1(x)| \le \frac{1}{2}|B_N(x) - A_N(x)| \overset{\text{def}}{=} C_N(x);$$
$$|T_1(x)| \le \frac{1}{2}|B_T(x) - A_T(x)| \overset{\text{def}}{=} C_T(x).$$

We obtain estimates of the values of σ_x, σ_y, and τ_{xy} by relying on the following easily proved proposition:

Lemma. *Let Q, f, A, and B be the functions of [2], which are in $L^\infty(W)$ and integrable over W, and let $A(x) \le f(x) \le B(x)$. Then*

$$\max_{f: A \le f \le B} \left| \int_W Q(x) f(x)\, dx \right| = |\gamma| + \int_W C(x)|Q(x)|\, dx, \tag{8}$$

where

$$C(x) = \frac{1}{2}[B(x) - A(x)], \quad \gamma = \frac{1}{2}\int_W [A(x) + B(x)]Q(x)\, dx.$$

Equality holds in (8) *at the functions*

$$f^*(x) = \frac{1}{2}[A(x) + B(x)] + C(x) \cdot \operatorname{sgn}(\gamma Q(x)) \ (\gamma \ne 0), \ f^*(x) = C(x) \cdot \operatorname{sgn} Q(x) \ (\gamma = 0). \tag{9}$$

If we apply the lemma to (7), we obtain

$$\max_{M_N, M_T} |\sigma_x| = \frac{1}{\pi h(\mu_2 - \mu_1)}[|\gamma_N| + |\gamma_T| + \int_U |Q_x(\xi; z)| C_N(\xi)\, d\xi + \int_V |R_x(\xi; z)| C_T(\xi)\, d\xi,$$

where

$$\gamma_N + \frac{1}{2}\int_U Q_x(\xi; z)[A_N(\xi) + B_N(\xi)]\, d\xi,$$
$$\gamma_T = \frac{1}{2}\int_V R_x(\xi; z)[A_T(\xi) + B_T(\xi)]\, d\xi.$$

In the case $\gamma_T \neq 0$, $\gamma_N \neq 0$ the largest value of σ_x is attained when

$$\begin{aligned} N_{\text{opt}}(x) &= \frac{1}{2}[A_N(x) + B_N(x)] + C_N(x)\operatorname{sgn}(\gamma_N Q_x(x;z)), \\ T_{\text{opt}}(x) &= \frac{1}{2}[A_T(x) + B_T(x)] + C_T(x)\operatorname{sgn}(\gamma_T R_x(x,z)). \end{aligned} \tag{10}$$

If $\gamma_T = 0$ and $\gamma_N = 0$, then

$$\begin{aligned} N_{\text{opt}}(x) &= C_N(x)\operatorname{sgn} Q_x(x;z), \\ T_{\text{opt}}(x) &= C_T(x)\operatorname{sgn} R_x(x,z). \end{aligned} \tag{11}$$

In the simplest case of (11) one should take account of the behavior of the functions $Q_1 := Q_x$ and $R_1 := R_x$:

$$Q_1(\xi;z) \leq 0 \ \forall \xi \in \mathbb{R}, \quad \operatorname{sgn} R_1(\xi;z) = \operatorname{sgn}(\xi - x) \ (x = \operatorname{Re} z) \forall x \in \mathbb{R}.$$

In particular if $U = (x - \alpha, x + \alpha)$ and $T(\xi) = 0$, then

$$\max_{|N| \leq l_N} |\sigma_x| = \frac{2\mu_1\mu_2 l_N}{\pi h(\mu_2 - \mu_1)}\left[\mu_2 \arctan\frac{\alpha}{\mu_2 y} - \mu_1 \arctan\frac{\alpha}{\mu_1 y}\right].$$

Similarly we obtain

$$\max_{|N| \leq l_N} |\sigma_y| = \frac{2l_N}{\pi h(\mu_2 - \mu_1)}\left[\mu_2 \arctan\frac{\alpha}{\mu_1 y} - \mu_2 \arctan\frac{\alpha}{\mu_2 y}\right], \quad \max_{|N| \leq l_N} |\tau_{xy}| = \frac{\mu_1\mu_2 l_N}{\pi h(\mu_2 - \mu_1)} \ln\frac{\mu_2^2(x^2 + \mu_1^2 y^2)}{\mu_1^2(\alpha^2 + \mu_2^2 y^2)}.$$

The optimal actions are $N_{\text{opt}}(\xi) \equiv l_N$ (for σ_x and σ_y) and $N_{\text{opt}}(\xi) = l_N \operatorname{sgn}(\xi - x)$ (for τ_{xy}).

The expansion of the sphere of activity of a bounded load $N(\xi)$ ($|N(\xi)| \leq l_N$, $|x - \xi| \leq \alpha$) for an increasing α leads to a monotone increase of the quantity $\max|\sigma_x|$. The limiting values are

$$\max|\sigma_x| = \frac{\mu_1\mu_2 l_N}{h}, \quad \max|\sigma_y| = \frac{l_N}{h}, \quad \max|\tau_{xy}| = \frac{2l_N}{\pi h(\mu_2 - \mu_1)} \ln\frac{\mu_2}{\mu_1}.$$

Suppose the rectifiable arc (L) is given by the equations

$$x = x(t), \quad y = y(t), \quad \alpha \leq t \leq \beta$$

and has length L. The average value

$$\bar{\sigma}_x = \frac{1}{L}\int_{(L)} \sigma_x(x,y)\,dl,$$

computed with respect to the boundary loads $N(x)$ and $T(x)$, is minimized using the lemma. Let $(L) : x = 0$, $y = t$ $(0 \leq t \leq H)$ under the restrictions (11). We find

$$\bar{\sigma}_x = \frac{1}{\pi HR(\mu_2 - \mu_1)} \int_{-\infty}^{\infty} \left[N(\xi)\frac{\mu_1\mu_2}{2} \ln\frac{\xi^2 + \mu_1^2 H^2}{\xi^2 + \mu_2^2 H^2} + T(\xi)\left(\mu_2 \arctan\frac{\mu_2 H}{\xi} - \mu_1 \arctan\frac{\mu_1 H}{\xi}\right)\right] d\xi.$$

The coefficient of $N(\xi)$ is nonnegative for all $\xi \in (-\infty, \infty)$ and the sign of the coefficient of $T(\xi)$ coincides with the sign of ξ. Therefore under the restrictions (11)

$$\max \bar{\sigma}_x = \frac{1}{\pi Hh(\mu_2 - \mu_1)}\left[\frac{\mu_1\mu_2}{2} \int_U \ln\frac{\xi^2 + \mu_2^2 H^2}{\xi^2 + \mu_1^2 H^2}\,d\xi + \int_V \left(\mu_2 \arctan\frac{\mu_2 H}{\xi} - \mu_1 \arctan\frac{\mu_1 H}{\xi}\right)\operatorname{sgn}\xi\,d\xi.\right.$$

The optimal loads are

$$N_{\text{opt}}(x) \equiv -l_N \ (x(U), \quad T_{\text{opt}}(x) = l_T \operatorname{sgn} \ (x \in V).$$

Now let $(L): x = t,\ y = 1\ (0 \le t \le H),\ 0 \le N(x) \le l_N(1-|x|),\ N(x) = 0\ (|x| > 1),\ T(x) \equiv 0$. With respect to

$$\bar{\tau}_{xy} = \frac{1}{H} \int\limits_{(L)} \tau_{xy}(x, y)\, dl$$

we obtain the following result

$$\bar{\tau}_{xy} = \frac{\mu_1\mu_2}{2\pi H h(\mu_2 - \mu_1)} \int_{-1}^{1} N(\xi) \ln \frac{[(\xi - H)^2 + \mu_2^2][\xi^2 + \mu_1^2]}{[(\xi - H)^2 + \mu_1^2][\xi^2 + \mu_2^2]}\, d\xi, \tag{12}$$

$$\max \bar{\tau}_{xy} = \frac{\mu_1\mu_2 l_N}{2\pi H h(\mu_2 - \mu_1)} \begin{cases} -\displaystyle\int_{-1}^{1} (1 - |\xi|) Q(\xi)\, d\xi, & H \ge 2, \\ \max\left(\displaystyle\int_{\frac{H}{2}}^{1} (1 - |\xi|) Q(\xi)\, d\xi, \int_{-1}^{\frac{H}{2}} (1 - |\xi|) Q(\xi)\, d\xi \right), & H < 2, \end{cases}$$

where $Q(\xi)$ is the coefficient of $N(\xi)$ in (12). The optimal function is

$$N_{\text{opt}}(x) = l_N \begin{cases} 1 - |x|, & x : \operatorname{sgn}(\gamma Q(x)) = +1, \\ 0, & x : \operatorname{sgn}(\gamma Q(x)) = -1 \text{ or } |x| \ge 1. \end{cases}$$

Here

$$\gamma = \frac{1}{2} \int_{-1}^{1} (1 - |x|) Q(x)\, dx,$$

and the sign of γ depends on the specific values of μ_1, μ_2, and H.

Using the example of optimizing σ_x at a point one can see the advantage of a pulsed boundary action in comparison with a ("smoother") integral action. We consider functions of the form

$$\begin{aligned} N_1(\xi) &= \sum_\nu d_\nu \delta(\xi - \xi_\nu), \quad \sum |d_\nu| \le l_N, \quad x - 1 \le \xi_\nu \le x + a, \\ T_1(\xi) &= \sum_\mu d_\mu \delta(\xi - \eta_\mu), \quad \sum |d_\mu| \le l_T, \quad x - a \le \eta_\mu \le x + a, \end{aligned} \tag{13}$$

and

$$N_2(\xi) : \int_{x-a}^{x+a} |N_2(\xi)|\, d\xi \le l_N, \quad T_2(\xi) : \int_{x-a}^{x+a} |T_2(\xi)|\, d\xi \le l_T. \tag{14}$$

In the case (13) we shall use the notation $N_1(\xi)\, d\xi = d\sigma_1(\xi)$, $T_1(\xi)\, d\xi = d\sigma_2(\xi)$. Here σ_1 and σ_2 are jump functions having bounded variation equal to the sum of the absolute values of the jumps. Relation (7) implies the inequality

$$|\sigma_x| \le \frac{1}{\pi h(\mu_2 - \mu_1)} [\max_{|\xi - x| \le a} |Q_1(\xi; z)| \operatorname{Var} \sigma_1(\xi) + \max_{|\xi - x| \le a} |R_1(\xi; z) \cdot \operatorname{Var} \sigma_2(\xi)|].$$

Equality is attained at elementary functions of the form (13) with a single jump at the point of absolute maximum of the function $Q_1(\xi; x)$ ($R_1(\xi; z)$ for the second term; one can also take two jumps symmetric about the point x). It is easy to see that $\xi_0 = x + y\sqrt{\mu_1\mu_2}$ is the point of absolute maximum of the function Q_1 and that $\xi_0' = x + a$ is the same point for the function R_1. Thus if $y\sqrt{\mu_1\mu_2} \le a$, then

$$\max_{N_1, T_1} |\sigma_x| = \frac{1}{\pi h} \left[\frac{\mu_1\mu_2 t_N}{y^2(\mu_1 + \mu_2)} + \frac{a^2(\mu_1 + \mu_2) l_T}{(a^1 + \mu_1^2 y^2)(a^2 + \mu_2^2 y^2)} \right]. \tag{15}$$

The optimal actions are

$$N_1(\xi) = l_N \delta(\xi - y\sqrt{\mu_1\mu_2}), \quad T_1(\xi) = l_T \delta(\xi - a).$$

In the case $y\sqrt{\mu_1\mu_2} > a$ the optimal actions are of the same type:

$$N_2(\xi) = l_N\delta(\xi - a), \quad T_2(\xi) = l_T\delta(\xi - a).$$

Consider functions of class (14). The following sharp inequalities hold:

$$|\sigma_x| \le \frac{1}{\pi h(\mu_2 - \mu_1)}\Bigg[\max|Q_1(\xi; z)| \cdot \int\limits_{|\xi - x| \le a} |N(\xi)|\,d\xi + \max|R_1(\xi; z)| \cdot \int\limits_{|\xi - x| \le a} |T(\xi)|\,d\xi$$

$$\le \frac{1}{\pi h(\mu_2 - \mu_1)}[l_N \cdot \max|Q_1| + l_T \cdot \max|R_1|].$$

It is easy to exhibit approximate identities $N_{2,x}(\xi)$, $T_{2,\nu}(\xi)$ of class (14) such that the values of $|\sigma_x|$ corresponding to them are monotone decreasing and tend to the value (15).

Literature Cited

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